Uniform Computable Categoricity and Function Composition

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Function composition and computability

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Definition (LG.)

The *composition spectrum* of a partial function h is the set

$$\mathrm{CompSpec}(h) := \{(\mathbf{a}, \mathbf{b}) : \exists f \leq_P \mathbf{a},\ \exists g \leq_P \mathbf{b},\ g \circ f = h\} \subseteq \mathcal{D}_P{}^2.$$

$\mathcal{D}_P$ are the *partial degrees*.

$(\mathbf{a}, \mathbf{b}) \in \mathrm{CompSpec}(h)$ means that we can split a computation for h into an $\mathbf{a}$-computable "preprocessing" step, followed by a $\mathbf{b}$-computable "postprocessing" step.

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- If $\mathbf{a} \geq_P h$ or $\mathbf{b} \geq_P h$, then $(\mathbf{a},\mathbf{b}) \in \mathrm{CompSpec}(h)$.

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- $\mathrm{CompSpec}(h)$ is closed upwards in each coordinate.
- If $\mathbf{a} \geq_P h$ or $\mathbf{b} \geq_P h$, then $(\mathbf{a}, \mathbf{b}) \in \mathrm{CompSpec}(h)$.
- If $(\mathbf{a}, \mathbf{b}) \in \mathrm{CompSpec}(h)$ then $\mathbf{a} \oplus \mathbf{b} \geq_P h$.

Warm-up: Splitting c.e. sets over composition

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Observation

For every c.e. set B, CompSpec($\emptyset'$) $\subseteq$ CompSpec(B).
That is, $\emptyset'$ is the hardest c.e. set to compute via composition.

Proof: Fix computable function α such that $\alpha(e, n) \in \emptyset'$ iff $n \in W_e$. Let $W_{e_0} = B$. If $g \circ f = \emptyset'$, then $\widehat{f}(n) = f(\alpha(e_0, n))$, $\widehat{g} = g$. Then $\widehat{g} \circ \widehat{f} = B$. □

Warm-up: Splitting c.e. sets over composition

Theorem (Sacks Splitting Theorem; Sacks '63)

If A is a noncomputable c.e. set, there are disjoint low c.e. sets X and Y such that $A = X \sqcup Y$ and $X, Y <_T A$.

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Corollary

If A is a noncomputable c.e. set, there are functions $f, g <_P A$ such that $A = g \circ f$. That is, $(\deg_P(f), \deg_P(g)) \in CompSpec(A)$.

Proof: Let $f(n) = \langle n, X(n) \rangle$. Let $g(\langle n, 1 \rangle) = 1$ and $g(\langle n, 0 \rangle) = Y(n)$. Then $f \equiv_P X$, $g \equiv_P Y$, and $g \circ f = A$. □

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So $\mathbf{a} \geq_P A$ or $\mathbf{b} \geq_P A$ are not necessary for $(\mathbf{a}, \mathbf{b}) \in \text{CompSpec}(A)$.

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Proposition (LG.)

For every c.e. set A, there are c.e. sets X and Y such that $A \leq_T X \oplus Y$, but for all $f \leq_T X$ and $g \leq_T Y$, $g \circ f \neq A$. That is, $(\deg_P(X), \deg_P(Y)) \notin \mathit{CompSpec}(A)$.

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Corollary

There are c.e. sets X and Y such that $X \oplus Y \equiv_T \emptyset'$, but $CompSpec(\emptyset') \subsetneq CompSpec(X \oplus Y)$.

Computable categoricity

Definition

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- $(\omega, <)$ is "uniformly" $\emptyset'$-computably categorical (but not computably categorical).
- (ω, S) is computably categorical, but not "uniformly" (we need $\mathbf{0}'$ to find where to map 0).

Weakly uniform computable categoricity

When the language is understood, we write $\mathcal{M}_e$ for the (partial) computable structure for which $\varphi_e = D(\mathcal{M}_e)$.

Definition (Kudinov '96; Downey-Hirschfeldt-Khoussainov '03)

A computable structure $\mathcal{A}$ is *weakly uniformly computably categorical (w.u.c.c.) if there is a Turing functional* Γ *such that for all e, if* $\mathcal{M}_e \cong \mathcal{A}$ *then* $\Phi_{\Gamma(e)} : \mathcal{A} \cong \mathcal{M}_e$.

Strong uniform categoricity requires uniformity in the *diagram* of a copy instead of the index (see Ventsov '92; Downey-Hirschfeldt-Khoussainov '03; relativization: R. Miller '17).

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- If $\mathcal{A}$ is *w.u.c.c.* then $\mathcal{A} + \mathcal{A} + \dots$ is computably categorical.
- There are two possible places to use oracles to relativize this.

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Definition (LG.)

A computable structure $\mathcal{A}$ is *weakly X-uniformly Y-computably categorical (w.X.u.Y.c.c.)* if there is a Turing functional Γ such that for all e, if $\mathcal{M}_e \cong \mathcal{A}$ then $\Phi^Y_{\Gamma^X(e)} : \mathcal{A} \cong \mathcal{M}_e$.

The *weakly uniform categoricity spectrum* of $\mathcal{A}$ is

$$\text{wuCatSpec}(\mathcal{A}) = \{(\mathbf{c}, \mathbf{d}) : \mathcal{A} \text{ is } w.\mathbf{c}.u.\mathbf{d}.c.c.\}.$$

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Observation

(ω, S) is $w.X.u.Y.c.c.$ iff there are partial functions $f \leq_T X$ and $g \leq_T Y$ such that if $\mathcal{M}_e \cong \mathcal{A}$ then $g(f(e)) = \min(\mathcal{M}_e)$.

Weak uniform categoricity and function composition

Observation

Trying to find $\min(\mathcal{M}_e)$ in an arbitrary computable copy $\mathcal{M}_e \cong (\omega, S)$ is like trying to compute a limit-computable (Δ^0_2) function: our guess may change many times, but is eventually constantly equal to the right answer.

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Proposition (LG.)

(ω, S) is $w.\mathbf{c}.u.\mathbf{d}.c.c.$ if and only if there exist partial functions $f \leq_T \mathbf{c}$ and $g \leq_T \mathbf{d}$ such that for all $e, x < \omega$,
$\Phi_e^{\emptyset'}(x) \downarrow m \implies g(f(\langle e, x\rangle)) \downarrow = m.$

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*(ω, S) is w.**c**.u.**d**.c.c. if and only if there exist partial functions $f \leq_T \mathbf{c}$ and $g \leq_T \mathbf{d}$ such that for all $e, x < \omega$, $\Phi_e^{\emptyset'}(x) \downarrow m \implies g(f(\langle e, x \rangle)) \downarrow = m$.*

$$\mathrm{CompSpec}(\emptyset') \supseteq \mathrm{wuCatSpec}((\omega, S)) \supseteq \mathrm{CompSpec}\left(\langle e, x \rangle \mapsto \Phi_e^{\emptyset'}(x)\right)$$

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Proposition (LG.)

There is a Σ_2^0 set $X \not\geq_T \emptyset'$ and a c.e. set $Y <_T \emptyset'$ such that (ω, S) is $w.X.u.Y.c.c.$

Questions

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Are there $X, Y <_T \emptyset'$ such that (ω, S) is w.X.u.Y.c.c.?

Question

How does the weak uniform categoricity of (ω, S) relate to that of other computable structures?

- *Hardest among structure that are w.u.$\emptyset'$.c.c. and w.u.c.c. "with parameters".*
- *Same as other natural examples: $\mathcal{R} \sqcup K_\omega$, finite joins, ...*

Question

What can be said about the relationship between weak uniform computable categoricity and strong uniform/relative computable categoricity?

Thank you!

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