Isomorphism Spectra and Computably Composite Structures

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Computable structures

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Remark

We may assume that all structures are relational.

Isomorphisms

Definition

If $\mathcal{A} \cong \mathcal{B}$ are computable structures, their *isomorphism spectrum* is the set of Turing degrees

$$\mathrm{IsoSpec}(\mathcal{A}, \mathcal{B}) = \{\mathbf{d} : (\exists f : \mathcal{A} \cong \mathcal{B})\ f \leq_T \mathbf{d}\}.$$

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Example

Let $A \subseteq \omega$ be a c.e. set with a fixed computable enumeration $(a_i : i < \omega)$. Consider the structure $(\omega, <_A)$ where

- $2n <_A 2m$ for all $n < m$,
- $2a_i <_A 2i + 1 <_A 2a_i + 2$ for all $i < \omega$.

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The (unique) isomorphism $f : (\omega, <) \cong (\omega, <_A)$ has $f \equiv_T A$. So, $\mathrm{IsoSpec}((\omega, <), (\omega, <_A)) = \mathcal{D}(\geq \deg_T(A)) := \{\mathbf{d} : \mathbf{d} \geq \deg_T(A)\}$, the *cone above* $\deg_T(A)$.

Computable categoricity

Definition

The *categoricity spectrum* of $\mathcal{M}$ is

$$\mathrm{CatSpec}(\mathcal{M}) = \bigcap_{\mathcal{A} \cong \mathcal{B} \cong \mathcal{M}} \mathrm{IsoSpec}(\mathcal{A}, \mathcal{B})$$

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Example

$\mathrm{CatSpec}(\omega, <) = \mathcal{D}(\geq \mathbf{0}')$.

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Theorem (following Bazhenov, Kalimullin, Yamaleev 2016 & 2020)

If $CatSpec(\mathcal{M}) = \mathcal{D}(\geq \mathbf{d})$ *and* $\mathbf{d}$ *is not the strong degree of categoricity of any structure, then there exist computable copies* $\mathcal{A} \cong \mathcal{B} \cong \mathcal{M}$ *such that* $IsoSpec(\mathcal{A}, \mathcal{B})$ *is not a finite union of cones.*

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Goal: Show that (computable) unions of isomorphism spectra are also isomorphism spectra.

Union of two isomorphism spectra

Given computable copies $\mathcal{A}_0 \cong \mathcal{B}_0$ and $\mathcal{A}_1 \cong \mathcal{B}_1$, we want to construct computable structures $\mathcal{M} \cong \mathcal{N}$ such that

$$\mathrm{IsoSpec}(\mathcal{M}, \mathcal{N}) = \mathrm{IsoSpec}(\mathcal{A}_0, \mathcal{B}_0) \cup \mathrm{IsoSpec}(\mathcal{A}_1, \mathcal{B}_1).$$

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"Attach" the original structures to P_4, the 4-vertex undirected path:

$$\mathcal{M}\colon \quad \mathcal{A}_0 - \mathcal{A}_1 - \mathcal{B}_1 - \mathcal{B}_0$$
$$\mathcal{N}\colon \quad \mathcal{A}_0 - \mathcal{B}_1 - \mathcal{A}_1 - \mathcal{B}_0$$

$\mathcal{M}$ and $\mathcal{N}$ are isomorphic and have two kinds isomorphisms, corresponding to the two automorphisms of P_4.

Computably composite structures

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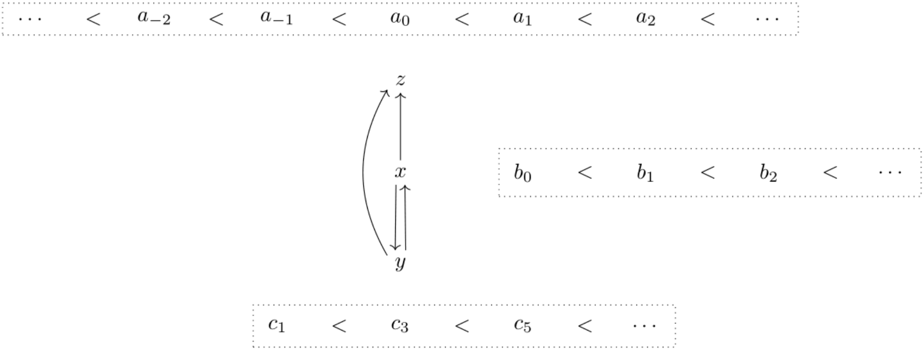

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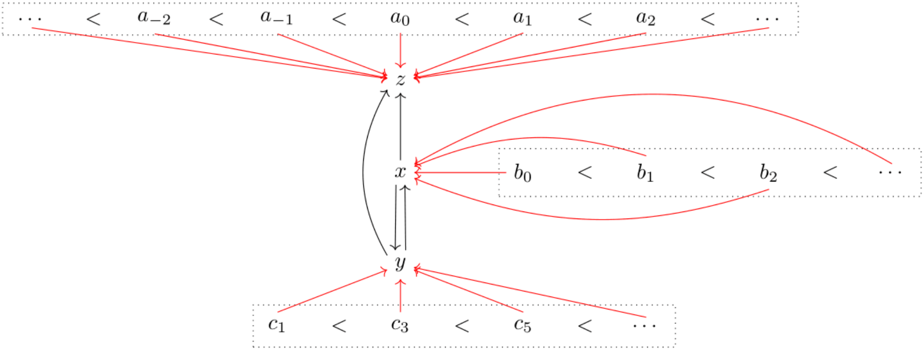

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Theorem

Suppose $\mathcal{G}[\mathbf{B}]$ is a computable copy of $\mathcal{S}[\mathbf{A}]$. Then the isomorphisms from $\mathcal{S}[\mathbf{A}]$ to $\mathcal{G}[\mathbf{B}]$ are exactly the maps of the form

$$\rho = \theta \cup \bigcup_{x \in S} \psi_x$$

where $\theta : \mathcal{S} \cong \mathcal{G}$ and $\psi_x : \mathcal{A}_x \cong \mathcal{B}_{\theta(x)}$ for each $x \in S$.

Union of two isomorphism spectra

$\mathcal{M} = P_4[\mathcal{A}_0, \mathcal{A}_1, \mathcal{B}_1, \mathcal{B}_0]$:

$\mathcal{A}_0 - \mathcal{A}_1 - \mathcal{B}_1 - \mathcal{B}_0$

$\mathcal{N} = P_4[\mathcal{A}_0, \mathcal{B}_1, \mathcal{A}_1, \mathcal{B}_0]$:

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The isomorphisms $\rho : \mathcal{M} \cong \mathcal{N}$ have the form

$$\rho = \theta \cup \bigcup_{x \in P_4} \psi_x$$

where $\theta : P_4 \cong P_4$ and ψ_x is an isomorphism between the components at index x in $\mathcal{M}$ and at index $\theta(x)$ in $\mathcal{N}$.

$$\mathrm{IsoSpec}(\mathcal{N}, \mathcal{M}) = \mathrm{IsoSpec}(\mathcal{A}_0, \mathcal{B}_0) \cup \mathrm{IsoSpec}(\mathcal{A}_1, \mathcal{B}_1).$$

$\mathcal{H}$

We define a structure, $\mathcal{H} = (H, \{D_i\}_{i<\omega}, \{E_i\}_{i<\omega})$ with universe $H = [\omega]^{<\omega} \cup (\omega \times \{0, 1\})$.

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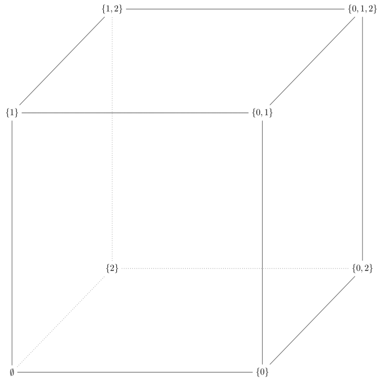
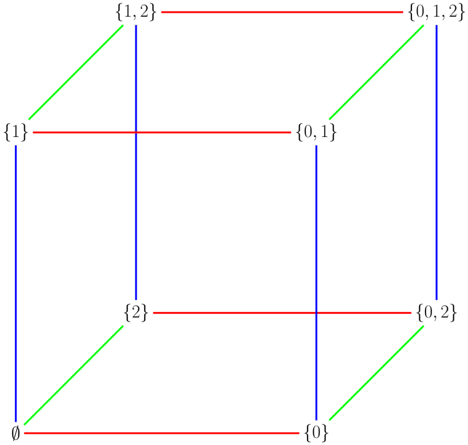


Think of $[\omega]^{<\omega}$ as an infinite-dimensional cube where $X, Y \in [\omega]^{<\omega}$ are adjacent if $|X \triangle Y| = 1$.

$\mathcal{H}$

For $X, Y \in [\omega]^{<\omega}$,

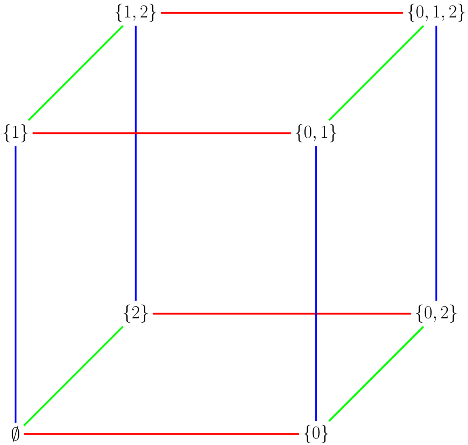
$$E_i(X, Y) \Leftrightarrow E_i(Y, X) \Leftrightarrow X \triangle Y = \{i\}.$$



$\mathcal{H}$

The two opposite "faces" of the cube $[\omega]^{<\omega}$ in the i^{th} dimension are the sets

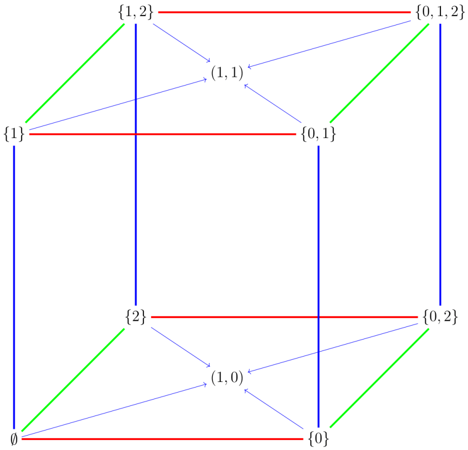
$$L_i = \{X \in [\omega]^{<\omega} : i \notin X\} \text{ and } R_i = \{X \in [\omega]^{<\omega} : i \in X\}.$$



$\mathcal{H}$

We label the faces L_i and R_i with the elements $(i, 0)$ and $(i, 1)$.

We also add directed D_i-edges from the points of L_i to $(i, 0)$ and from the points of R_i to $(i, 1)$.



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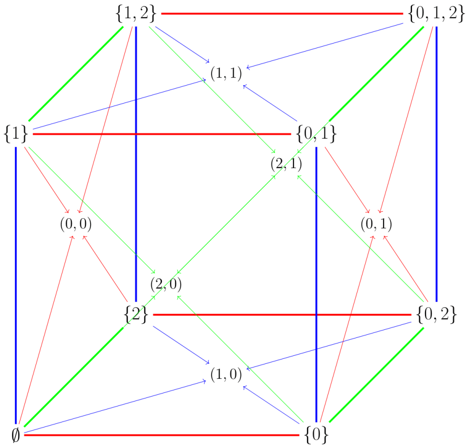
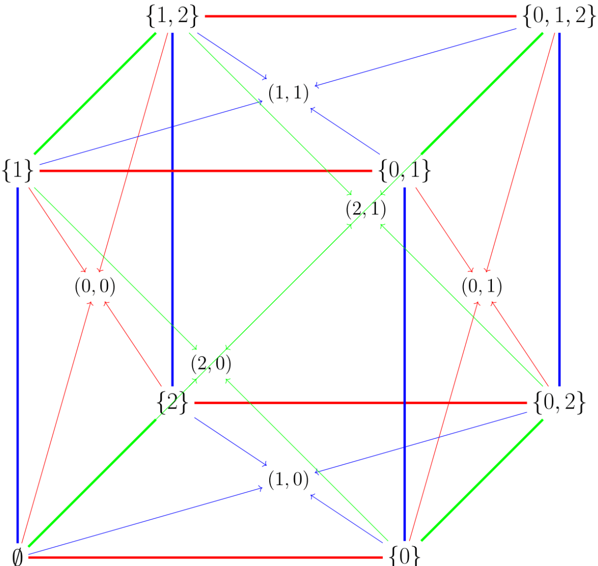


Automorphisms of $\mathcal{H}$

Let h_X be the unique automorphism of $\mathcal{H}$, with $h_X(\emptyset) = X$.



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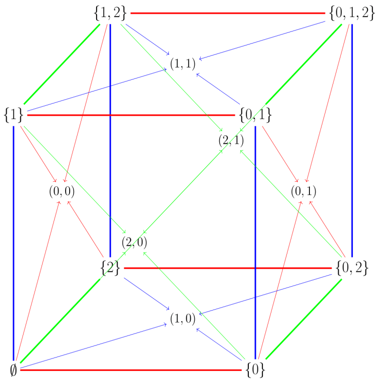

$$h_X(Y) = X \triangle Y \text{ if } Y \in [\omega]^{<\omega}$$

$$h_X(i, a) = \begin{cases} (i, a) & \text{if } i \notin X \\ (i, 1 - a) & \text{if } i \in X \end{cases}$$

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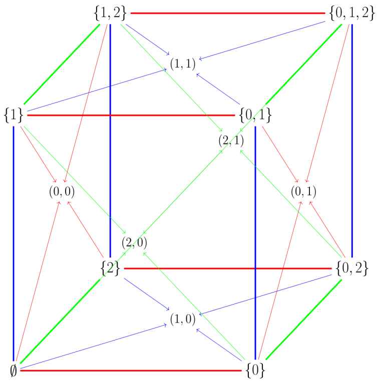

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Theorem

The automorphisms of $\mathcal{H}$ are exactly $\{h_X : X \in [\omega]^{<\omega}\}$. Moreover, every isomorphism between computable copies of $\mathcal{H}$ is computable.

Computably composite structures on $\mathcal{H}$

Theorem

Given any two uniformly computable collections of copies $\mathbf{A} = \{\mathcal{A}_i : i < \omega\}$ *and* $\mathbf{B} = \{\mathcal{B}_i : i < \omega\}$ *such that for each* i, $\mathcal{A}_i \cong \mathcal{B}_i$, *there exists a structure with two computable copies* $\mathcal{M} \cong \mathcal{N}$ *where* $IsoSpec(\mathcal{M}, \mathcal{N}) = \bigcup_{i<\omega} IsoSpec(\mathcal{A}_i, \mathcal{B}_i)$.

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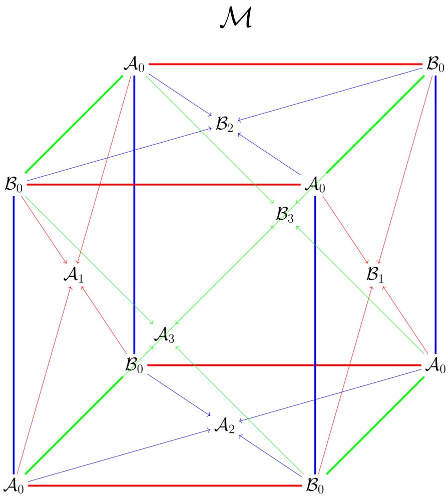

Isomorphism spectrum that is not a finite union of cones

By a result of Thomason (1971), there is a uniformly c.e. sequence of sets $\{Z_i : i < \omega\}$ such that if $i \neq j$, then Z_i and Z_j are Turing incomparable.

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So the collection $\mathbf{B} = \{(\omega, <_{Z_i})\}_{i<\omega}$ is uniformly computable. Recall that $\mathrm{IsoSpec}((\omega, <), (\omega, <_{Z_i})) = \mathcal{D}(\geq \deg_T(Z_i))$.

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Let $\mathcal{M}$ and $\mathcal{N}$ be the computably composite structures on $\mathcal{H}$ corresponding to $\mathbf{A} = \{(\omega, <)\}_{i<\omega}$ and $\mathbf{B}$. Then,

$$\begin{aligned}\mathrm{IsoSpec}(\mathcal{M}, \mathcal{N}) &= \bigcup_{i<\omega} \mathrm{IsoSpec}((\omega, <), (\omega, <_{Z_i})) \\ &= \{\mathbf{d} : (\exists i < \omega)\mathbf{d} \geq \deg(Z_i)\}\end{aligned}$$

which is not a finite union of cones.

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We always have

$$\mathrm{CatSpec}(\mathcal{H}[\mathbf{A}]) \subseteq \bigcap_{i<\omega} \mathrm{CatSpec}(\mathcal{A}_i).$$

The other inclusion holds if and only if all $\mathcal{A}_i$ satisfy a particular notion of *uniform categoricity*.