Higher dimension partition principles in uncountable Hausdorff spaces

Joey Lakerdas-Gayle

University of Waterloo

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Galvin's problem and Polish grids

Definition (Galvin's problem in dimension d)

G_d is the statement: There is a bound $N < \omega$ such that for every finite colouring $c : [\mathbb{R}]^d \to k$, there exists $Y \subseteq \mathbb{R}$ homeomorphic to $\mathbb{Q}$ such that $\left|c\left[[Y]^d\right]\right| \leq N$.

Inamdar (2024) proved that G_2 is true with $N = 2$.

Definition (Polish grid principle in dimension d)

PG_d is the statement: For every sequence $(X_0, \ldots, X_{d-1})$ of perfect polish spaces and every finite colouring $\gamma : \prod_{i<d} X_i \to k$, there exists somewhere dense subspaces $Y_i \subseteq X_i$ such that $\prod_{i<d} Y_i$ is monochromatic.

Galvin's problem and Polish grids

Theorem (Raghavan & Todorcevic, 2023)

$2^{\aleph_0} \leq \aleph_{d-2} \implies \neg \mathsf{G}_d.$

Theorem (Lambie-Hanson & Zucker, 2024)

$2^{\aleph_0} \leq \aleph_{d-2} \implies \neg \mathsf{PG}_d.$

Goal: Find a family of partition principle (parametrized by dimension) that holds in every model of $2^{\aleph_0} \leq \aleph_{d-2}$ and implies $\neg \mathsf{G}_d$ and $\neg \mathsf{PG}_d$.

Unordered products

Definition

For sets $A_0, \ldots, A_{d-1}$, define the *unordered product*
$\circledast_{i<d} A_i := \{\{a_0, \ldots, a_{d-1}\} : a_i \in A_i, i \neq j \Rightarrow a_i \neq a_j\} \subseteq [\bigcup_{i<d} A_i]^d$

Definition

$\mathrm{B}_d^{\circledast}(\kappa)$ is the statement: For any Hausdorff space X with $|X| = \kappa$, there is a colouring $c : [X]^d \to \omega$ such that if $A_0, \ldots, A_{d-1} \subseteq X$ and each A_i is homeomorphic to $\mathbb{Q}$, then $c\left[\circledast_{i<d} A_i\right] = \omega$.

Theorem

$n \leq d - 2 \implies \mathrm{B}_d^{\circledast}(\aleph_n)$.

Taking $A_0 = \cdots = A_{d-1}$ and $X = \mathbb{R}$, we get $\mathrm{B}_d^{\circledast}(2^{\aleph_0}) \implies \neg \mathrm{G}_d$.

So $2^{\aleph_0} \leq \aleph_{d-2} \implies \mathrm{B}_d^{\circledast}(2^{\aleph_0}) \implies \neg \mathrm{G}_d$.

$\mathrm{B}_d^{\circledast}(2^{\aleph_0}) \implies \neg\mathrm{PG}_d$

$\mathrm{B}_d^{\circledast}(\kappa)$

For any Hausdorff space X with $|X| = \kappa$, there is $c : [X]^d \to \omega$ such that if $A_i \subseteq X$ are homeomorphic to $\mathbb{Q}$, then $c\left[\circledast_{i<d} A_i\right] = \omega$.

PG_d

For every sequence $(X_0, \ldots, X_{d-1})$ of perfect polish spaces and every finite colouring $\gamma : \prod_{i<d} X_i \to k$, there exists somewhere dense subspaces $Y_i \subseteq X_i$ such that $\prod_{i<d} Y_i$ is monochromatic.

1. Apply $\mathrm{B}_d^{\circledast}(2^{\aleph_0})$ to $X = \bigsqcup_{i<d} X_i$ to get $c : [X]^d \to \omega$.
2. If $Y_i \subseteq X_i$ is somewhere dense, then there is $A_i \subseteq Y_i$ homeomorphic to $\mathbb{Q}$ and $c[\circledast_{i<d} A_i] = \omega$.
3. Define $c' : \prod_{i<d} X_i \to \omega$ by $c'((x_0, \ldots, x_{d-1})) = c(\{x_0, \ldots, x_{d-1}\})$.

So $2^{\aleph_0} \leq \aleph_{d-2} \implies \mathrm{B}_d^{\circledast}(2^{\aleph_0}) \implies \neg\mathrm{PG}_d$.

Summary

Question

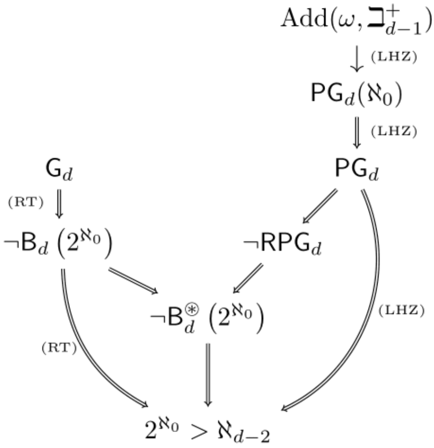
Is G_d consistent for any $d \geq 3$?

Question

Is it consistent that $2^{\aleph_0} > \aleph_{d-2}$ but $\neg\mathsf{PG}_d$?

Question

Is it consistent that $\neg\mathsf{G_d}$ and $\neg\mathsf{B}_d$? That $\neg\mathsf{PG}_d$ and $\neg\mathsf{RPG}_d$?



Thank you!

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