

Relativizing uniform computable categoricity

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Computable categoricity

Definition

A computable structure $\mathcal{A}$ is *X-computably categorical (X.c.c.)* for a set $X \subseteq \mathbb{N}$ if for every computable copy $\mathcal{B}$ of $\mathcal{A}$, there is an X -computable isomorphism from $\mathcal{A}$ to $\mathcal{B}$.

The *categoricity spectrum* of $\mathcal{A}$ is the set

$$\text{CatSpec}(\mathcal{A}) = \{\mathbf{d} : \mathcal{A} \text{ is } \mathbf{d}.\text{c.c.}\}$$

Two kinds of uniformity

Uniformity in the diagram:

Definition (Ventsov '92; Downey–Hirschfeldt–Khoussainov '03)

A computable structure $\mathcal{A}$ is *uniformly computably categorical* (u.c.c.) if there is a Turing functional Ψ such that if $\mathcal{B}$ is a computable copy of $\mathcal{A}$, then $\Psi^{D(\mathcal{B})} : \mathcal{A} \cong \mathcal{B}$.

Uniformity in the index:

Definition (Kudinov '96; Downey–Hirschfeldt–Khoussainov '03)

A computable structure $\mathcal{A}$ is *weakly uniformly computably categorical* (w.u.c.c.) if there is a partial computable function f such that $\forall e \in \mathbb{N}, \mathcal{M}_e \cong \mathcal{A} \implies \Phi_{f(e)} : \mathcal{A} \cong \mathcal{M}_e$.

- $(\omega, S, 0)$ is u.c.c., $(\omega, <)$ and (ω, S) are not w.u.c.c.
- Kudinov constructs a w.u.c.c. structure that is not u.c.c.

Relativizing uniform categoricity

Definition (R. Miller 2017)

A countable structure $\mathcal{A}$ is X -uniformly $\Delta_{1+\alpha}$ -computably categorical if there is a Turing functional Ψ such that if $\mathcal{B} \cong \mathcal{C} \cong \mathcal{A}$, then $\Psi^{X \oplus \mathcal{B}^{(\alpha)} \oplus \mathcal{C}^{(\alpha)}} : \mathcal{B} \cong \mathcal{C}$.

Theorem (R. Miller 2017; Ventsov 1992)

A countable structure $\mathcal{A}$ is $X.u.\Delta_{1+\alpha}.c.c.$ if and only if $\mathcal{A}$ has a X -c.e. **Scott family** of $\Sigma_{\alpha+1}^c$ -formulas without parameters.

(I.e., every $\bar{a} \in \mathcal{A}$ satisfies some $\phi \in S$, and if $\bar{a}, \bar{b}$ satisfy the same $\phi \in S$ then $(\mathcal{A}, \bar{a}) \cong (\mathcal{A}, \bar{b})$)

Due to Ash–Knight–Manasse–Slaman '89 and Chisholm '90, $\mathcal{A}$ is relatively $\Delta_{\alpha+1}^0.c.c.$ iff $(\mathcal{A}, \bar{a})$ is $u.\Delta_{1+\alpha}.c.c.$ for some $\bar{a} \in \mathcal{A}$.

Relativizing uniform categoricity

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- $(\omega, <)$ and (ω, S) are $u.\Delta_2$.c.c., but not X .u.c.c. for any X (they are not u.c.c. on any cone).
- Fokina–Kalimullin–Miller (2010) construct a computable structure $\deg\text{Cat}(\mathcal{M}_X) = \deg_T(X)$ for any 2-c.e. X .
 $\mathcal{M}_X$ is X .u.c.c. (it is u.c.c. on the cone above X)

Relativizing weak uniform categoricity

Definition (L.)

A computable structure $\mathcal{A}$ is *weakly X -uniformly Y -computably categorical* ($w.X.u.Y.c.c.$) if there is a partial X -computable function f such that $\forall e \in \mathbb{N}, \mathcal{M}_e \cong \mathcal{A} \implies \Phi_{f(e)}^Y : \mathcal{A} \cong \mathcal{M}_e$.

If $\mathcal{A}$ is Y -c.c. then $\{X : \mathcal{A} \text{ is } w.X.u.Y.c.c.\}$ says how hard it is to find indices for Y -oracle isomorphisms.

Weak uniform categoricity

$\mathcal{A}$ is w.X.u.Y.c.c. iff

$$(\exists f \leq_T X)(\forall e) \mathcal{A} \cong \mathcal{M}_e \implies \Phi_{f(e)}^Y : \mathcal{A} \cong \mathcal{M}_e.$$

(ω, S) is Y -computably categorical for every Y .

How hard is it to *find* indices for Y -oracle isomorphisms?

Weak uniform categoricity

$\mathcal{A}$ is w.X.u.Y.c.c. iff

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(ω, S) is Y -computably categorical for every Y .

How hard is it to *find* indices for Y -oracle isomorphisms?

- (ω, S) is w. $\emptyset'$.u.c.c. and w.u. $\emptyset'$.c.c.,
- If (ω, S) is w.X.u.Y.c.c. then $X \oplus Y \geq_T \emptyset'$.

Weak uniform categoricity

$\mathcal{A}$ is $w.X.u.Y.c.c.$ iff

$$(\exists f \leq_T X)(\forall e) \mathcal{A} \cong \mathcal{M}_e \implies \Phi_{f(e)}^Y : \mathcal{A} \cong \mathcal{M}_e.$$

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How hard is it to *find* indices for Y -oracle isomorphisms?

- (ω, S) is $w.\emptyset'.u.c.c.$ and $w.u.\emptyset'.c.c.$,
- If (ω, S) is $w.X.u.Y.c.c.$ then $X \oplus Y \geq_T \emptyset'$.

But $X \oplus Y \geq_T \emptyset'$ is not enough:

Proposition (L.)

(ω, S) is $w.X.u.Y.c.c.$ iff $\exists$ partial functions $f \leq_T X$ and $g \leq_T Y$ such that for all e, n, m , $\Phi_e^{\emptyset'}(n) \downarrow = m \implies g(f(e, n)) = m$.

In particular, $X' \oplus Y' \geq_T \{\langle e, n \rangle : \Phi_e^{\emptyset'}(n) \downarrow\} \geq_T \emptyset''$.

Simulating convergent $\emptyset'$ problems

Theorem (L.)

(ω, S) is $w.X.u.Y.c.c.$ for some $\Pi_2^0 X \not\leq_T \emptyset'$ and $\Pi_1^0 Y <_T \emptyset'$.

Theorem (L., J. Miller, M. Soskova)

(ω, S) is not $w.X.u.Y.c.c.$ for any $X <_T \emptyset'$ and $Y \not\leq_T \emptyset'$.

Turing degrees $\mathbf{d} < \mathbf{0}'$ are no better than $\mathbf{0}$ at finding indices for isomorphisms between computable copies of (ω, S) .

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Other structures with the same weakly uniform categoricity:

- $\{\text{random graph}\} \sqcup \{\text{complete graph on } \omega\}$
- Every linear order, injection structure, equivalence structure that is computably categorical, but not without parameters (Goncharov–Dzgoev, Remmel, Cenzer–Harizanov–Remmel, Calvert–Cenzer–Harizanov–Morozov)

Weakly X -uniform computable categoricity

Question

Given a computably categorical structure $\mathcal{A}$, what is $\{X : \mathcal{A} \text{ is w.}X\text{.u.c.c.}\}$?

Examples

- (ω, S) is w. X .u.c.c. iff $X \geq_T \emptyset'$.
- [A. Miller] $(\mathbb{Q}, <, \frac{1}{n} : n \geq 1)$ in the language $L = (<, c_n : n \geq 1)$ is w. X .u.c.c. iff $X \geq_T \emptyset''$.
- “ $\varphi_e : \mathcal{M}_i \cong \mathcal{M}_j$ ” is Π_2^0 , so every computably categorical structure is w. $\emptyset''$.u.c.c.

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Proposition (L.)

There is a computably categorical structure that is not w. $\emptyset'$.u.c.c., but is w.X.u.c.c. for some Π_2^0 set $X \not\geq_T \emptyset'$.

Future work

Question

Is there a computably categorical structure $\mathcal{A}$ that is not w.u.c.c., but is w. X .u.c.c. for some $X <_T \emptyset'$?

Future work

Question

Is there a computably categorical structure $\mathcal{A}$ that is not $w.u.c.c.$, but is $w.X.u.c.c.$ for some $X <_T \emptyset'$?

Question

For which $Y <_T \emptyset'$ is there an $X \not\leq_T \emptyset'$ such that (ω, S) is $w.X.u.Y.c.c.$?

Future work

Question

Is there a computably categorical structure $\mathcal{A}$ that is not w.u.c.c., but is w.X.u.c.c. for some $X <_T \emptyset'$?

Question

For which $Y <_T \emptyset'$ is there an $X \not\leq_T \emptyset'$ such that (ω, S) is w.X.u.Y.c.c.?

Question

Is there a noncomputable partial function h such that whenever $g \circ f \supseteq h$, $\text{graph}(f) \geq_e \text{graph}(h)$ or $\text{graph}(g) \geq_e \text{graph}(h)$?

Thank you!

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