Isomorphism Spectra and Uniform Computable Categoricity

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Computable categoricity

Definition

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Definition (Fokina, Kalimullin, R. Miller, 2010)

The *categoricity spectrum* of a computable structure $\mathcal{A}$ is $\mathrm{CatSpec}(\mathcal{A}) = \{\deg_T(X) : \mathcal{A} \text{ is } X\text{-computably categorical}\}$.

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If $\mathrm{CatSpec}(\mathcal{A})$ has a least element, $\mathbf{d}$, then $\mathbf{d}$ is the *degree of categoricity* of $\mathcal{A}$.

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E.g., $(\mathbb{Q}, <)$ is computably categorical, $\mathrm{degCat}(\omega, <) = \mathbf{0}'$.

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The *isomorphism spectrum* of a pair of computable structures $\mathcal{A}$ and $\mathcal{B}$ is $\mathrm{IsoSpec}(\mathcal{A}, \mathcal{B}) = \{\mathbf{d} : (\exists f : \mathcal{A} \cong \mathcal{B})\ \mathbf{d} \geq_T f\}$.

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Question (Fokina, Kalimullin, R. Miller, 2010)

Is every degree of categoricity a strong degree of categoricity?

Degrees of categoricity

Theorem (Bazhenov-Kalimullin-Yamaleev 2020)

If $\mathcal{M}$ has degree of categoricity $\mathbf{d}$ and there is a finite family $\{\mathcal{A}_i, \mathcal{B}_i : i < k\}$ of computable copies of $\mathcal{M}$ such that

$$CatSpec(\mathcal{M}) = \bigcap_{i<k} IsoSpec(\mathcal{A}_i, \mathcal{B}_i)$$

then the direct product $\mathcal{M} \times \mathcal{M} \times \cdots \times \mathcal{M}$ (k times) is a computable structure with strong degree of categoricity $\mathbf{d}$.

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So if $\mathbf{d} = \text{degCat}(\mathcal{M})$ is not a strong degree of categoricity, then there is no such finite family. That is, $\mathcal{M}$ has *infinite spectral dimension*.

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A structure of Turetsky (2020) has infinite spectral dimension and a degree of categoricity.

Infinite spectral dimension

Suppose $\text{degCat}(\mathcal{M}) = \mathbf{d}$ and there is a computable sequence $(\mathcal{A}_i, \mathcal{B}_i)_{i<\omega}$ of computable copies of $\mathcal{M}$ such that

$$\mathrm{CatSpec}(\mathcal{M}) = \bigcap_{i<\omega} \mathrm{IsoSpec}(\mathcal{A}_i, \mathcal{B}_i).$$

Does the infinite direct product $\mathcal{M} \times \mathcal{M} \times \mathcal{M} \times \dots$ have strong degree of categoricity $\mathbf{d}$?

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No. E.g. (ω, S) is computably categorical, but $(\omega, S) \times (\omega, S) \times (\omega, S) \times$ has strong degree of categoricity $\mathbf{0}'$.

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Observation

If $\mathcal{M}$ is "*uniformly* $\mathbf{d}$-computably categorical", then $\mathcal{M} \times \mathcal{M} \times \mathcal{M} \times \ldots$ is $\mathbf{d}$-computably categorical.

E.g., $(\mathbb{Q}, <) \times (\mathbb{Q}, <) \times (\mathbb{Q}, <) \times \ldots$ is computably categorical.

Unions of isomorphism spectra

The class of isomorphism spectra is closed under computable union:

Proposition (L.)

If $(\mathcal{A}_i, \mathcal{B}_i)_{i<\omega}$ *is a uniformly computable sequence of computable copies of structures, then there are computable structures* $\mathcal{M}$ *and* $\mathcal{N}$ *such that* $IsoSpec(\mathcal{M}, \mathcal{N}) = \bigcup_{i<\omega} IsoSpec(\mathcal{A}_i, \mathcal{B}_i)$.

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Observation

If $(\mathcal{A}_i, \mathcal{B}_i)_{i<\omega}$ are "uniformly **d**-computably categorical", then $\mathcal{M}$ is **d**-computably categorical.

Uniform computable categoricity

Definition (Ventsov '92; Downey-Hirschfeldt-Khoussainov '03)

A computable structure $\mathcal{A}$ is *(strongly) uniformly computably categorical (u.c.c.)* if there is a Turing functional Ψ such that if $\mathcal{B}$ is a computable copy of $\mathcal{A}$, then $\Psi^{D(\mathcal{B})} : \mathcal{A} \cong \mathcal{B}$.

Definition (Kudinov '96; Downey-Hirschfeldt-Khoussainov '03)

A computable structure $\mathcal{A}$ is *weakly uniformly computably categorical (w.u.c.c.) if there is a Turing functional* Γ *such that whenever* $\varphi_e = D(\mathcal{B})$ *and* $\mathcal{B} \cong \mathcal{A}$*, then* $\Phi_{\Gamma(e)} : \mathcal{A} \cong \mathcal{B}$.

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- A structure of Kudinov (1997) is *w.u.c.c.* but not *u.c.c.*
- If $\mathcal{M}$ is *w.u.c.c.*, then $\mathcal{M} \times \mathcal{M} \times \ldots$ is computably categorical.

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Definition (R. Miller, 2017)

A countable structure $\mathcal{A}$ is *X-uniformly $\Delta_{1+\alpha}$-computably categorical* if there is a Turing functional Ψ such that if $\mathcal{B} \cong \mathcal{C} \cong \mathcal{A}$, then $\Psi^{X \oplus \mathcal{B}^{(\alpha)} \oplus \mathcal{C}^{(\alpha)}} : \mathcal{B} \cong \mathcal{C}$.

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Definition (L.)

A computable structure $\mathcal{A}$ is *weakly X-uniformly Y-computably categorical ($w.X.u.Y.c.c.$)* if there is a Turing functional Γ such that whenever $\varphi_e = D(\mathcal{B})$ and $\mathcal{B} \cong \mathcal{A}$, then $\Phi^{Y}_{\Gamma^X(e)} : \mathcal{A} \cong \mathcal{B}$.

Weakly uniform computable categoricity

Examples:

- $(\mathbb{Q}, <)$ is $w.\emptyset.u.\emptyset.c.c.$ (so, $w.X.u.Y.c.c.$ for all $X, Y \subseteq \mathbb{N}$).

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- $(\omega, <)$ is $w.X.u.Y.c.c.$ if and only if $Y \geq \emptyset'$.
- (ω, S) is not $w.\emptyset.u.\emptyset.c.c.$, but it is $w.\emptyset'.u.\emptyset.c.c.$ and $w.\emptyset.u.\emptyset'.c.c.$

If $f : (\omega, S) \cong \mathcal{M}_e = (\omega, \widetilde{S})$, then $f(n) = \widetilde{S}^n(\min(\mathcal{M}_e))$.

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- If (ω, S) is $w.X.u.Y.c.c.$, then $X \oplus Y \geq_T \emptyset'$.
  Q: Is this sufficient?

Answer: No (to both questions).

$X \oplus Y \geq_T \emptyset'$ is not sufficient

Proposition (L.)

There is a pair of disjoint c.e. sets X, Y such that $X \oplus Y \equiv_T \emptyset'$ but (ω, S) is not w.X.u.Y.c.c.

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Proof idea.

Finite injury priority construction:

- Build c.e. sets $X \sqcup Y = \emptyset'$ (so that $X \oplus Y \geq_T \emptyset'$)

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- Build c.e. sets $X \sqcup Y = \emptyset'$ (so that $X \oplus Y \geq_T \emptyset'$)
- Build copies $\mathcal{M}_{e_j} \cong (\omega, S)$ such that $\Phi^Y_{\Phi^X_j(e_j)}(0) \neq \min(\mathcal{M}_{e_j})$ for each j

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Corollary

$w.(A \oplus B).u.Y.c.c. \not\Rightarrow w.A.u.(B \oplus Y).c.c.$

$X \geq_T \emptyset'$ or $Y \geq_T \emptyset'$ is not necessary

Proposition (L.)

There is a Σ^0_2 set $X \not\geq_T \emptyset'$ and a c.e. set $Y <_T \emptyset'$ such that (ω, S) is $w.X.u.Y.c.c.$

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$R_e: \ \mathcal{M}_e \cong (\omega, S) \Rightarrow \Phi^Y_{\Psi^X(e)} = \min(\mathcal{M}_e)$

$N^X_e: \ \Phi^X_e \neq C$ for some c.e. set C

$N^Y_e: \ \Phi^Y_e \neq D$ for some c.e. set D

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- To satisfy R_e, watch $D(\mathcal{M}_e)$ and enumerate into X or Y when the guess for $\min(\mathcal{M}_e)$ changes.

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- To satisfy R_e, watch $D(\mathcal{M}_e)$ and enumerate into X or Y when the guess for $\min(\mathcal{M}_e)$ changes.
- **Problem:** We might restrict initial segments of X and Y such that we cannot satisfy R_e.

$X \geq_T \emptyset'$ or $Y \geq_T \emptyset'$ is not necessary

The proof is a priority construction on a tree of strategies:

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- **New problem:** Different parts of the tree are ordered differently and may enumerate into X and Y "incorrectly".
- **(Partial) solution:** Allow elements to be removed from the approximation of X. Hence, X is Σ_2. The asymmetry of R_e allows Y to remain c.e.

Questions

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Are there $X, Y <_T \emptyset'$ such that (ω, S) is w.X.u.Y.c.c.?

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Which sets of pairs of degrees have the form

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for some computable structure $\mathcal{A}$? Which pairs are minimums of these sets?

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Question

What can be said about the relationship between weak uniform computable categoricity and strong uniform/relative computable categoricity?

Thank you!

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